A Circulant Formulation of the Napoleon-Douglas-Neumann Theorem

Geng-zhe Chang* and Philip J. Davis
Division of Applied Mathematics
Brown University
Providence, Rhode Island 02912

Submitted by Richard A. Brualdi

ABSTRACT

This paper places the Napoleon-Douglas-Neumann theorem within the theory of circulant matrices. The "symmetrical components" introduced by Neumann are similarly treated.

1. INTRODUCTION

Triangle ABC is given arbitrarily (Figure 1). Denote its sides and area by a, b, c, and Δ respectively. Based upon each side of the triangle, erect outwardly (inwardly) an isosceles triangle with the base angle $\pi/6$. Denote the free vertices—the vertices other than A, B, C—by A_0, B_0, C_0 (A_1, B_1, C_1). Simple geometric considerations lead to

$$\overline{C_0A_0}^2 = \tfrac{1}{6}(a^2+b^2+c^2)+\tfrac{2}{3}\sqrt{3}\,\Delta \tag{1}$$

and similarly

$$\overline{C_1A_1}^2 = \tfrac{1}{6}(a^2+b^2+c^2)-\tfrac{2}{3}\sqrt{3}\,\Delta. \tag{2}$$

The following three conclusions come out immediately from (1) and (2):

(i) Triangles $A_0B_0C_0$ and $A_1B_1C_1$ are equilateral.
(ii) $\Delta = \Delta_0 - \Delta_1$, where

$$\Delta_0 = \text{area of } A_0B_0C_0,$$

$$\Delta_1 = \text{area of } A_1B_1C_1.$$

*On leave from Department of Mathematics, University of Science and Technology of China, Hefei, Anhui, China.

52 Vanderbilt Ave., New York, NY 10017 0024-3795/83/$3.00

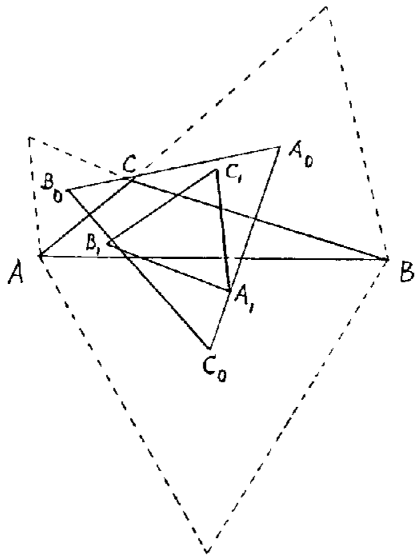

FIG. 1.

(iii) The sum of the squared sides of ABC
$=$ the sum of the squared sides of $A_0B_0C_0$
$+$ the sum of the squared sides of $A_1B_1C_1$.

All this constitutes Napoleon's theorem. This theorem provides the construction of two equilateral triangles from an arbitrarily given triangle by constructing isosceles triangles on the sides of the given triangle.

This has been generalized to arbitrary polygons in the plane. The generalization of (i) was carried out independently and almost simultaneously by Jesse Douglas [6, 7] and B. H. Neumann [8]. Roughly speaking, for an arbitrarily given n-gon P, $n-1$ regular n-gons $P_1, P_2, \ldots, P_{n-1}$ can be constructed by constructing finitely many isosceles triangles.

Numerous proofs of this generalization have been given, e.g., [1], [2], [9], [10]. In [8], Neumann generalized (ii) and (iii) to the polygonal case in terms of "symmetrical components." In [10], Neumann revisited these facts by means of the linear algebra of finite-dimensional vector spaces. In this paper we go slightly farther as far as theorems are concerned, and show how the generalized Napoleon's theorem can be placed neatly within the context of the theory of circulant matrices.

2. PRELIMINARIES

Let the complex numbers $z_1, z_2, \ldots, z_n$ denote the ordered vertices of a polygon P, and set $P = [z_1, z_2, \ldots, z_n]^\tau$, where τ indicates the matrix transpose operation. Let C be a complex number. Now on each side of P erect a triangle which is directly similar to the triangle $0, 1, c$. The n free vertices so obtained, forming a new n-gon, can be expressed [2] by the matrix multiplication $[(1-c)I + c\Pi]P$, where I is the $n \times n$ identity matrix and

$$\Pi = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

It is well known [3–5] that Π can be diagonalized by the Fourier matrix F. More precisely, we have

$$\Pi = F^*\Omega F, \tag{3}$$

where

$$F^* = \left[\omega^{(j-1)(k-1)}\right]/\sqrt{n}, \tag{4}$$

$$\Omega = \operatorname{diag}\left(1, \omega, \omega^2, \ldots, \omega^{n-1}\right), \tag{5}$$

$$\omega = \exp(2\pi i/n), \tag{6}$$

and * denotes the conjugate transpose operation. It is clear that F is unitary,

$$F^* = F^{-1} \tag{7}$$

and that

$$F = \overline{F^*}. \tag{8}$$

Consider $n-1$ matrices $K_j = (1-c_j)I + c_j\Pi$, where complex numbers $c_1, c_2, \ldots, c_{n-1}$ are to be determined. From (3) and (7) we see that

$$K_j = F^*\Lambda_j F \tag{9}$$

where

$$\Lambda_j = (1 - c_j)I + c_j\Omega \tag{10}$$

is a diagonal matrix with $(1,1)$ element equal to 1, for $j = 1, 2, \ldots, n-1$. We now choose c_j in such a way that the $(j+1, j+1)$ element in Λ_j becomes equal to zero, i.e., $(1 - c_j) + c_j\omega^j = 0$. Thus

$$c_j = \frac{1}{1 - \omega^j}, \qquad j = 1, 2, \ldots, n-1. \tag{11}$$

The matrices $K_1, K_2, \ldots, K_{n-1}$ are now well defined.

Forming the all possible products of $n-2$ matrices from $K_1, K_2, \ldots, K_{n-1}$:

$$N_\nu = K_1 \cdots K_{\nu-1}K_{\nu+1} \cdots K_{n-1}, \qquad \nu = 1, 2, \ldots, n-1, \tag{12}$$

we have

$$N_\nu = F^* D_\nu F, \tag{13}$$

where $D_\nu = \Lambda_1 \cdots \Lambda_{\nu-1}\Lambda_{\nu+1} \cdots \Lambda_{n-1}$ is a diagonal matrix with

$$(1,1) \text{ element} = 1,$$

$$(\nu+1, \nu+1) \text{ element} = \prod_{\substack{k=1 \\ k \neq \nu}}^{n-1} \frac{\omega^\nu - \omega^k}{1 - \omega^k}$$

$$= \prod_{\substack{k=1 \\ k \neq \nu}}^{n-1} \frac{\omega^\nu(1 - \omega^{k-\nu})}{1 - \omega^k}$$

$$= \omega^{\nu(n-2)} \frac{1 - \omega^\nu}{1 - \omega^{n-\nu}} = -\frac{1}{\omega^\nu},$$

and zero everywhere else. Hence we can write

$$D_\nu = E_1 - \frac{E_{\nu+1}}{\omega^\nu}, \tag{14}$$

where E_j designates the $n \times n$ matrix with 1 in (j, j) position and zero everywhere else, $j = 1, 2, \ldots, n$.

Inserting (14) into (13), we get the following decomposition of N_ν:

$$N_\nu = F^*E_1F - \frac{F^*E_{\nu+1}F}{\omega^\nu}. \tag{15}$$

It is obvious from (4) and (8) that

$$F^*E_{\nu+1}F = \frac{1}{n}\begin{bmatrix} 1 & & & & \\ & \omega^\nu & & & \\ & & \omega^{2\nu} & & \\ & & & \ddots & \\ & & & & \omega^{(n-1)\nu} \end{bmatrix}\begin{bmatrix} \omega^{-\nu} & \omega^{-2\nu} & \cdots & \omega^{-n\nu} \\ \omega^{-\nu} & \omega^{-2\nu} & \cdots & \omega^{-n\nu} \\ \cdots & \cdots & \cdots & \cdots \\ \omega^{-\nu} & \omega^{-2\nu} & \cdots & \omega^{-n\nu} \end{bmatrix},$$

$$\nu = 0, 1, 2, \ldots, n-1. \tag{16}$$

Consider the n-gon $P_\nu \equiv N_\nu P$, $\nu = 1, 2, \ldots, n-1$. By (15) and (16) we have

$$P_\nu = JP - \left[a_\nu, \omega^\nu a_\nu, \ldots, \omega^{(n-1)\nu}a_\nu\right]^\tau, \tag{17}$$

where

$$J = \frac{1}{n}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

and

$$a_\nu = \frac{1}{n}\sum_{k=1}^{n} \omega^{-k\nu} z_k, \qquad \nu = 1, 2, \ldots, n-1. \tag{18}$$

Since

$$JP = \left(\frac{z_1 + z_2 + \cdots + z_n}{n}\right)[1, 1, \ldots, 1]^\tau,$$

it is clear from (17) that P_ν is a regular n-gon of ω^ν type, i.e., its consecutive vertices occur at an equal angular interval of $2\pi\nu/n$ upon a circumference centered at $\tilde{z}=(z_1+z_2+\cdots+z_n)/n$, the centroid of the original polygon P.

We note that with c_j given in terms of (11), the triangle $0,1,c_j$ is isosceles with base angle $\pi/2-j\pi/n$. Recalling the definition of N_ν [see (12)] and the roles played by K_j, we have shown the following Napoleon-Douglas-Neumann constructions.

If isosceles triangles with base angles $\pi/2-j\pi/n$ are erected on the sides of an arbitrary polygon P, and if this process is repeated with the polygon formed by the free vertices of the triangles, but with a different value of j, and so on until all values $j=1,2,\ldots,n-1$ except ν have been used in arbitrary order, then a regular n-gon P_ν of ω^ν type is obtained. Its centroid coincides with that of the vertices of P.

We would like to point out that the second term on the right-hand side of (17) is the νth symmetrical component of P, introduced by Neumann.

3. FURTHER RESULTS

Since $E_1^2=E_1$, $E_{\nu+1}^2=E_{\nu+1}$, and $E_1E_{\nu+1}=0$ for $\nu>0$, then we have by (15) that

$$\begin{aligned} N_\nu^*N_\nu &= F^*\left(E_1-\frac{1}{\overline{\omega}^\nu}E_{\nu+1}\right)\left(E_1-\frac{1}{\omega^\nu}E_{\nu+1}\right)F \\ &= F^*(E_1+E_{\nu+1})F. \end{aligned}$$

Thus

$$\begin{aligned} \sum_{\nu=1}^{n-1} N_\nu^*N_\nu &= F^*[I+(n-2)E_1]F \\ &= I+(n-2)F^*E_1F. \end{aligned}$$

Hence by (16),

$$\sum_{\nu=1}^{n-1} N_\nu^*N_\nu = I+(n-2)J. \tag{19}$$

Assume that Q is an $n\times n$ circulant matrix, and consider the quadratic form associated with Q: P^*QP. Since N_ν and Q are circulants and all circulants commute,

$$\begin{aligned}\sum_{\nu=1}^{n-1} P_\nu^* Q P_\nu &= \sum_{\nu=1}^{n-1} P^* N_\nu^* Q N_\nu P \\ &= \sum_{\nu=1}^{n-1} P^* N_\nu^* N_\nu Q P = P^* \left(\sum_{\nu=1}^{n-1} N_\nu^* N_\nu \right) QP \\ &= P^*[I + (n-2)J]QP \\ &= P^*QP + (n-2)P^*JQP. \end{aligned} \tag{20}$$

In particular, if $JQ = 0$, then we see that

$$P^*QP = \sum_{\nu=1}^{n-1} P_\nu^* Q P_\nu. \tag{21}$$

Three special cases:

(i) The moment of inertia of n-gon P about its centroid is associated with a quadratic form whose matrix Q is given by (see [4], [5])

$$\begin{aligned} Q &= (I-J)^*(I-J) = (I-J)^2 \\ &= I - 2J + J^2 = I - 2J + J = I - J. \end{aligned}$$

It is clear that $Q = I - J$ is a circulant and

$$JQ = J(I-J) = J - J^2 = J - J = 0.$$

Hence we have by (21) that the moment of inertia of P about its centroid $\tilde{z}$ is equal to the sum of the moments of inertia of $P_1, P_2, \ldots, P_{n-1}$ about their own centroids (also $\tilde{z}$).

(ii) *Signed area.* In this case select Q as the circulant [4, 5]

$$Q = \frac{\Pi - \Pi^*}{4i} = \frac{\Pi - \Pi^T}{4i},$$

$$JQ = \frac{1}{4i}(J\Pi - J\Pi^T) = \frac{1}{4i}(J - J) = 0.$$

Hence we have by (21) that the sum of the signed areas of $P_1, P_2, \dots, P_{n-1}$ is just the signed area of P.

(iii) *Sum of squared sides.* Now select [4, 5]

$$Q = 2I - \Pi - \Pi^* = 2I - \Pi - \Pi^{\tau}.$$

Q is a circulant and

$$JQ = 2J - J\Pi - J\Pi^{\tau} = 2J - J - J = 0.$$

Then, again by (21) the sum of the squared sides is

$$\sum_{\nu=1}^{n-1} (\text{sum of the squared sides of } P_\nu).$$

One last word. Let Q be a circulant, and let s be the sum of the elements of the first row of Q. Then, it is clear that

$$JQ = QJ = sJ,$$

so that $JQ = 0$ if and only if $s = 0$. Then, by (20),

$$\begin{aligned}\sum_{\nu=1}^{n-1} P_\nu^* Q P_\nu &= P^*QP + (n-2)P^*JQP \\ &= P^*QP + \frac{n-2}{n} s|z_1 + z_2 + \cdots + z_n|^2 \\ &= P^*QP + n(n-2)s|\tilde{z}|^2.\end{aligned}$$

The second term of the right-hand side is zero if and only if either $s = 0$ or $\tilde{z} = 0$. Thus, if the original n-gon is such that $\tilde{z} = 0$ (its c.g. is at the origin), we have the decomposition theorem (21) for any circulant quadratic form. The condition $s = 0$, which is equivalent to the condition $JQ = 0$ and which we have exploited above, is itself equivalent to the translation invariance of the quadratic form P^*QP.

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